Kosmos, Friday March 10, 2023

# Euclid Saves Us from Ignorance
(according to Proclus)

Welcome to this Kosmos session on Euclid! I am including some materials for you to look over before we meet next week. These include some Euclid in Greek and English, as well as an "appreciation" of Euclid by the much later writer Proclus.

If you know some Greek, you won't find Euclid's Greek too difficult, although he does have his particular use of technical terms that you may be unfamiliar with.

I am a big fan of Euclid, and look forward to discussing him with you! Please feel free to bring any questions and comments to our session!

Best wishes,

Graeme Bird.

**Proclus Diadochus'** *Commentary on* Euclid'*s Elements*

(https://mathshistory.st-andrews.ac.uk/Extras/Proclus_history_geometry/)

Euclid, who was not much younger than Hermotimus and Philippus, composed Elements, putting in order many of the theorems of Eudoxus, perfecting many that had been worked on by Theaetetus, and furnishing with rigorous proofs propositions that had been demonstrated less rigorously by his predecessors. Euclid lived in the time of the first Ptolemy, for Archimedes, whose life overlapped the reign of this Ptolemy too, mentions Euclid. Furthermore, there is a story that Ptolemy once asked Euclid whether there was any shorter way to a knowledge of geometry than by the study of the *Elements*. Whereupon Euclid answered that there was no royal road to geometry. He is, then, younger than Plato's pupils and older than Eratosthenes and Archimedes, who, as Eratosthenes somewhere remarks, were contemporaries.

By choice Euclid was a follower of Plato and connected with this school of philosophy. In fact he set up as the goal of the Elements as a whole the construction of the so-called Platonic figures.

There are, in addition, many other mathematical works by Euclid, written with remarkable accuracy and scientific insight, such as the *Optics,* the *Catoptrics,* works on the *Elements of Music,* and the book *On Divisions.* But he is most to be admired for his *Elements of Geometry* because of the choice and arrangement of the theorems and problems made with regard to the elements. For he did not include all that he might have included, but only those theorems and problems which could fulfil the functions of elements .... If you seek to add or subtract anything, are you not unwittingly cast adrift from science and carried away toward falsehood and ignorance?

# Introduction to Euclid[§]

## Book 1 Definitions

### Ὅροι.

α΄. Σημεῖόν ἐστιν, οὗ μέρος οὐθέν.

β΄. Γραμμὴ δὲ μῆκος ἀπλατές.

γ΄. Γραμμῆς δὲ πέρατα σημεῖα.

δ΄. Εὐθεῖα γραμμή ἐστιν, ἥτις ἐξ ἴσου τοῖς ἐφ᾽ ἑαυτῆς σημείοις κεῖται.

ε΄. Ἐπιφάνεια δέ ἐστιν, ὃ μῆκος καὶ πλάτος μόνον ἔχει.

ϛ΄. Ἐπιφανείας δὲ πέρατα γραμμαί.

ζ΄. Ἐπίπεδος ἐπιφάνειά ἐστιν, ἥτις ἐξ ἴσου ταῖς ἐφ᾽ ἑαυτῆς εὐθείαις κεῖται.

η΄. Ἐπίπεδος δὲ γωνία ἐστὶν ἡ ἐν ἐπιπέδῳ δύο γραμμῶν ἁπτομένων ἀλλήλων καὶ μὴ ἐπ᾽ εὐθείας κειμένων πρὸς ἀλλήλας τῶν γραμμῶν κλίσις.

θ΄. Ὅταν δὲ αἱ περιέχουσαι τὴν γωνίαν γραμμαὶ εὐθεῖαι ὦσιν, εὐθύγραμμος καλεῖται ἡ γωνία.

ι΄. Ὅταν δὲ εὐθεῖα ἐπ᾽ εὐθεῖαν σταθεῖσα τὰς ἐφεξῆς γωνίας ἴσας ἀλλήλαις ποιῇ, ὀρθὴ ἑκατέρα τῶν ἴσων γωνιῶν ἐστι, καὶ ἡ ἐφεστηκυῖα εὐθεῖα κάθετος καλεῖται, ἐφ᾽ ἣν ἐφέστηκεν.

### Definitions

1. A point is that of which there is no part.

2. And a line is a length without breadth.

3. And the extremities of a line are points.

4. A straight-line is (any) one which lies evenly with points on itself.

5. And a surface is that which has length and breadth only.

6. And the extremities of a surface are lines.

7. A plane surface is (any) one which lies evenly with the straight-lines on itself.

8. And a plane angle is the inclination of the lines to one another, when two lines in a plane meet one another, and are not lying in a straight-line.

9. And when the lines containing the angle are straight then the angle is called rectilinear.

10. And when a straight-line stood upon (another) straight-line makes adjacent angles (which are) equal to one another, each of the equal angles is a right-angle, and the former straight-line is called a perpendicular to that upon which it stands.

§ EUCLID'S ELEMENTS OF GEOMETRY: The Greek text of J.L. Heiberg (1883–1885) from Euclidis Elementa, edidit et Latine interpretatus est I.L. Heiberg, in aedibus B.G. Teubneri, 1883–1885 edited, and provided with a modern English translation, by Richard Fitzpatrick

11. An obtuse angle is one greater than a right-angle.

12. And an acute angle (is) one less than a right-angle.

13. A boundary is that which is the extremity of something.

14. A figure is that which is contained by some boundary or boundaries.

15. A circle is a plane figure contained by a single line [which is called a circumference], (such that) all of the straight-lines radiating towards [the circumference] from one point amongst those lying inside the figure are equal to one another.

16. And the point is called the center of the circle.

17. And a diameter of the circle is any straight-line, being drawn through the center, and terminated in each direction by the circumference of the circle. (And) any such (straight-line) also cuts the circle in half.[†]

18. And a semi-circle is the figure contained by the diameter and the circumference cuts off by it. And the center of the semi-circle is the same (point) as (the center of) the circle.

19. Rectilinear figures are those (figures) contained by straight-lines: trilateral figures being those contained by three straight-lines, quadrilateral by four, and multilateral by more than four.

20. And of the trilateral figures: an equilateral triangle is that having three equal sides, an isosceles (triangle) that having only two equal sides, and a scalene (triangle) that having three unequal sides.

20 And of the trilateral figures: an equilateral triangle is that having three equal sides, an isosceles (triangle) that having only two equal sides, and a scalene (triangle) that having three unequal sides.

21 And further of the trilateral figures: a right-angled triangle is that having a right-angle, an obtuse-angled (triangle) that having an obtuse angle, and an acute-angled (triangle) that having three acute angles.

22 And of the quadrilateral figures: a square is that which is right-angled and equilateral, a rectangle that which is right-angled but not equilateral, a rhombus that which is equilateral but not right-angled, and a rhomboid that having opposite sides and angles equal to one another which is neither right-angled nor equilateral. And let quadrilateral figures besides these be called trapezia.

23 Parallel lines are straight-lines which, being in the same plane, and being produced to infinity in each direction, meet with one another in neither (of these directions).

## Postulates

1 Let it have been postulated to draw a straight-line from any point to any point.

2 And to produce a finite straight-line continuously in a straight-line.

3 And to draw a circle with any center and radius.

4 And that all right-angles are equal to one another.

5 And that if a straight-line falling across two (other) straight-lines makes internal angles on the same side (of itself) less than two right-angles, being produced to infinity, the two (other) straight-lines meet on that side (of the original straight-line) that the (internal angles) are less than two right-angles (and do not meet on the other side).[2]

## Common Notions

1 Things equal to the same thing are also equal to one another.

2 And if equal things are added to equal things then the wholes are equal.

3 And if equal things are subtracted from equal things then the remainders are equal.[3]

4 And things coinciding with one another are equal to one another.

5 And the whole [is] greater than the part.

[2]This postulate effectively specifies that we are dealing with the geometry of *flat*, rather than curved, space.

[3]As an obvious extension of C.N.s 2 & 3—if equal things are added or subtracted from the two sides of an inequality then the inequality remains an inequality of the same type.

# Euclid's 5$^{th}$ Postulate

(αἴτημα εʹ)

5. And that if a straight-line falling across two (other) straight-lines makes internal angles on the same side (of itself whose sum is) less than two right-angles, then the two (other) straight-lines, being produced to infinity (ἐπ᾽ ἄπειρον "boundless, limitless"), meet on that side (of the original straight-line) that the (sum of the internal angles) is less than two right-angles (and do not meet on the other side).‡

‡ This postulate effectively specifies that we are dealing with the geometry of *flat*, rather than curved, space.

# Definition of the Greek term ἄπειρος

i) boundless, infinite, σκότος Pi.Fr. 130.8;

ii) of number, countless, πλῆθος Hdt.1.204; ἀριθμὸς ἄ. πλήθει Pl.Prm.144a;

iii) εἰς ἄ. τὴν ἀδικίαν αὐξάνειν Id.Lg.910b;

iv) χρόνος ἄ. OGI383.113 (i B.C.):

v) τὸ ἄ. the Infinite, as a first principle, Arist.Ph.203a3, etc.; esp. in the system of Anaximander;

vi) in Trag., freq. of garments, etc., in which one is entangled past escape, i.e. without outlet, ἀμφίβληστρον A.Ag.1382;

# How to end a proof…

If you were attempting to prove that something is true, you would end by saying:

ὅπερ ἔδει δεῖξαι

"(Which is) the very thing it was required to show."

In Latin, "Q.E.D." (quod erat demonstrandum = "(the thing) that was to be demonstrated")

# How to end a proof…

If you were attempting to <u>construct something</u>, you would end by saying:

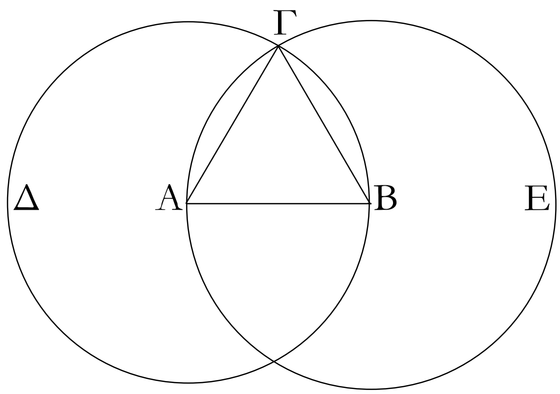
ὅπερ ἔδει <u>ποιῆσαι</u>

“(Which is) the very thing it was required <u>to do</u>/<u>make</u>.”

In Latin, “Q.E.F.” (quod erat faciendum= “(the thing) that was to be done/made”)

# ΣΤΟΙΧΕΙΩΝ α΄

## α΄




Ἐπὶ τῆς δοθείσης εὐθείας πεπερασμένης τρίγωνον ἰσόπλευρον συστήσασθαι.

Ἔστω ἡ δοθεῖσα εὐθεῖα πεπερασμένη ἡ ΑΒ.

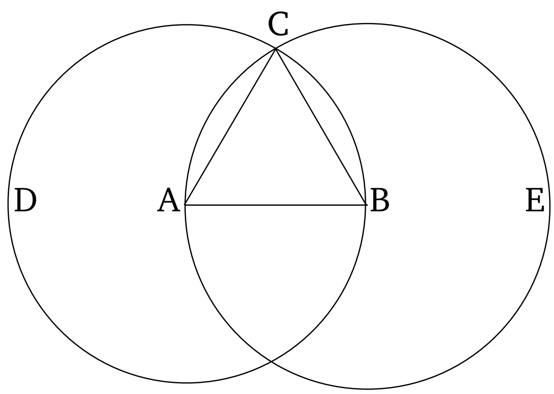
Δεῖ δὴ ἐπὶ τῆς ΑΒ εὐθείας τρίγωνον ἰσόπλευρον συστήσασθαι.

Κέντρῳ μὲν τῷ Α διαστήματι δὲ τῷ ΑΒ κύκλος γεγράφθω ὁ ΒΓΔ, καὶ πάλιν κέντρῳ μὲν τῷ Β διαστήματι δὲ τῷ ΒΑ κύκλος γεγράφθω ὁ ΑΓΕ, καὶ ἀπὸ τοῦ Γ σημείου, καθ᾽ ὃ τέμνουσιν ἀλλήλους οἱ κύκλοι, ἐπί τὰ Α, Β σημεῖα ἐπεζεύχθωσαν εὐθεῖαι αἱ ΓΑ, ΓΒ.

Καὶ ἐπεὶ τὸ Α σημεῖον κέντρον ἐστὶ τοῦ ΓΔΒ κύκλου, ἴση ἐστὶν ἡ ΑΓ τῇ ΑΒ· πάλιν, ἐπεὶ τὸ Β σημεῖον κέντρον ἐστὶ τοῦ ΓΑΕ κύκλου, ἴση ἐστὶν ἡ ΒΓ τῇ ΒΑ. ἐδείχθη δὲ καὶ ἡ ΓΑ τῇ ΑΒ ἴση· ἑκατέρα ἄρα τῶν ΓΑ, ΓΒ τῇ ΑΒ ἐστιν ἴση. τὰ δὲ τῷ αὐτῷ ἴσα καὶ ἀλλήλοις ἐστὶν ἴσα· καὶ ἡ ΓΑ ἄρα τῇ ΓΒ ἐστιν ἴση· αἱ τρεῖς ἄρα αἱ ΓΑ, ΑΒ, ΒΓ ἴσαι ἀλλήλαις εἰσίν.

Ἰσόπλευρον ἄρα ἐστὶ τὸ ΑΒΓ τρίγωνον. καὶ συνέσταται ἐπὶ τῆς δοθείσης εὐθείας πεπερασμένης τῆς ΑΒ· ὅπερ ἔδει ποιῆσαι.

## Proposition 1




To construct an equilateral triangle on a given finite straight-line.

Let $AB$ be the given finite straight-line.

So it is required to construct an equilateral triangle on the straight-line $AB$.

Let the circle $BCD$ with center $A$ and radius $AB$ have been drawn [Post. 3], and again let the circle $ACE$ with center $B$ and radius $BA$ have been drawn [Post. 3]. And let the straight-lines $CA$ and $CB$ have been joined from the point $C$, where the circles cut one another,[4] to the points $A$ and $B$ (respectively) [Post. 1].

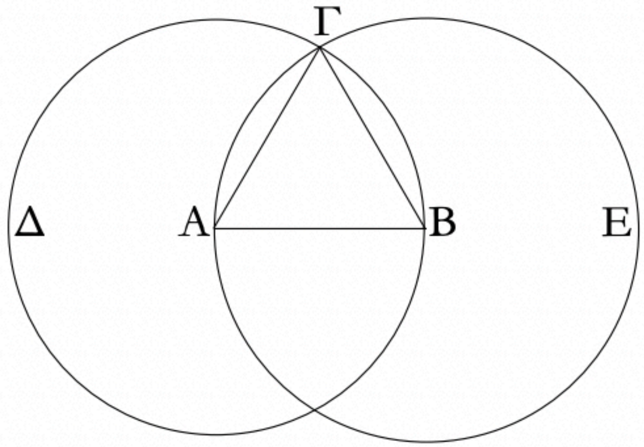
And since the point $A$ is the center of the circle $CDB$, $AC$ is equal to $AB$ [Def. 1.15]. Again, since the point $B$ is the center of the circle $CAE$, $BC$ is equal to $BA$ [Def. 1.15]. But $CA$ was also shown (to be) equal to $AB$. Thus, $CA$ and $CB$ are each equal to $AB$. But things equal to the same thing are also equal to one another [C.N. 1]. Thus, $CA$ is also equal to $CB$. Thus, the three (straight-lines) $CA$, $AB$, and $BC$ are equal to one another.

Thus, the triangle $ABC$ is equilateral, and has been constructed on the given finite straight-line $AB$. (Which is) the very thing it was required to do.

[4] The assumption that the circles do indeed cut one another should be counted as an additional postulate. There is also an implicit assumption that two straight-lines cannot share a common segment.

## α′.

Ἐπὶ τῆς δοθείσης εὐθείας πεπερασμένης τρίγωνον ἰσόπλευρον συστήσασθαι.

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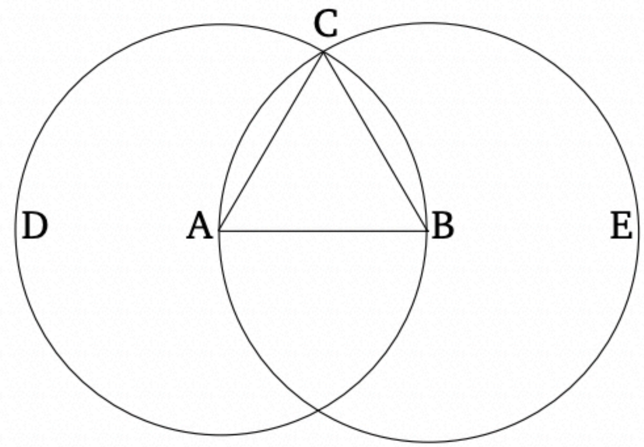
## Proposition 1

To construct an equilateral triangle on a given finite straight-line.

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